Europ. J. Combinatorics (1989) **10,** 551–560

The Associahedron and Triangulations of the n-gon

CARL W. LEE

Let P_n be a convex n-gon in the plane, $n \geqslant 3$. Consider Σ_n, the collection of all sets of mutually non-crossing diagonals of P_n. Then Σ_n is a simplicial complex of dimension $n-4$. We prove that Σ_n is isomorphic to the boundary complex of some $(n-3)$-dimensional simplicial convex polytope, and that this polytope can be geometrically realized to have the dihedral group D_n as its group of symmetries. Formulas for the f-vector and h-vector of this polytope and some implications for related combinatorial problems are discussed.

1. INTRODUCTION

Let P_n be a convex n-gon in the plane, $n \geqslant 3$. Apart from the n edges of P_n, the n-gon has $\binom{n}{2} - n = n(n-3)/2$ diagonals. Two different diagonals are said to *cross* if they intersect at a point other than, possibly, a common endpoint. Consider Σ_n, the collection of all sets of mutually non-crossing diagonals. The maximum size of such a set is $n-3$. We may therefore regard Σ_n as a simplicial complex of dimension $n-4$, having $n(n-3)/2$ vertices.

Perles [12] asked whether Σ_n is isomorphic to the boundary complex of some $(n-3)$-dimensional simplicial polytope. He cited Huguet and Tamari [8] in which a related polytopal object was discussed. Because maximum sets in Σ_n correspond to triangulations of P_n, we seek an $(n-3)$-dimensional polytope Q_n with one vertex for each diagonal of P_n and one facet for each triangulation of P_n. In this paper we show that such a polytope exists. We then consider formulas for the f-vector and h-vector of this polytope, and discuss some implications for related combinatorial problems, which we list at the end of Section 6.

Haiman [7] independently solved Perles' problem by constructing the dual of the desired Q_n, obtaining a defining set of inequalities, one for each diagonal of the n-gon. Because of the correspondence between triangulations of the n-gon and ways of parenthesizing a sequence of $n-1$ symbols, we will adopt Haiman's designation and refer to any polytope combinatorially equivalent to Q_n as the $(n-3)$-dimensional *associahedron*. Recall that the number of triangulations of the n-gon, and hence the number of facets of Q_n, is the $(n-1)$st Catalan number

$$c_{n-1} = \frac{1}{n-1}\binom{2n-4}{n-2}, \qquad n \geqslant 2.$$

See Gardner [5] for a pleasant introduction to this often-encountered sequence.

2. SIMPLICIAL COMPLEXES

For convenience we review some properties of simplicial complexes. A *simplicial complex* Δ is a non-empty collection of subsets of a finite set V with the property that $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$. For $F \in \Delta$ we say F is a *face* of Δ and the *dimension* of F, $\dim F$, equals $(\operatorname{card} F) - 1$. The *dimension* of Δ, $\dim \Delta$, is defined to be $\max\{\dim F : F \in \Delta\}$. Faces of Δ of dimension 0, 1, $(\dim \Delta) - 1$ and $\dim \Delta$ are called *vertices, edges, subfacets* and *facets* of Δ, respectively. For any finite set F, the set of all subsets of F will be denoted $\bar{F}$, and the set of all proper subsets of F will be denoted $\partial\bar{F}$. We will write $v_1 v_2 \cdots v_k$ as an abbreviation for the set $\{v_1, v_2, \ldots, v_k\}$ and will write $\bar{v}$ as an abbreviation for $\overline{\{v\}}$.

0195-6698/89/060551 + 10 \$02.00/0

Let Δ be a simplicial complex. If $F \in \Delta$, the *link* of F in Δ is the simplicial complex $\text{lk}_\Delta F = \{G \in \Delta: G \cap F = \varnothing, G \cup F \in \Delta\}$. If $F \neq \varnothing$, the *deletion* of F from Δ is the simplicial complex $\Delta \backslash F = \{G \in \Delta: F \nsubseteq G\}$.

Let Δ_1 and Δ_2 be simplicial complexes with disjoint sets of vertices. The *join* of Δ_1 and Δ_2 is the simplicial complex $\Delta_1 \cdot \Delta_2 = \{F_1 \cup F_2: F_1 \in \Delta_1, F_2 \in \Delta_2\}$. Suppose $F \neq \varnothing$ is a face of a simplicial complex Δ. Then the *stellar subdivision* of F in Δ is the simplicial complex $\text{st}(v, F)[\Delta] = (\Delta \backslash F) \cup (\bar{v} \cdot \partial \bar{F} \cdot \text{lk}_\Delta F)$, where v is a new vertex that is not a vertex of Δ. Note that during a stellar subdivision, the only old faces of Δ that are lost are those containing F, and the only new ones that are created are those containing v. We also observe that if F itself is a vertex; then $\text{st}(v, F)[\Delta]$ is isomorphic to Δ, the vertex F simply being relabeled.

If a simplicial complex Δ is *polytopal,* i.e. if Δ is isomorphic to the boundary complex $\Sigma(P)$ of some simplicial convex polytope P, then so is $\text{st}(v, F)[\Delta]$ for any $\varnothing \neq F \in \Delta$. One can, for example, choose a point v just 'above' the centroid of the face of P corresponding to F, and form the polytope $Q = \text{conv}(P \cup \{v\})$, where 'conv' means *convex hull.* Then $\text{st}(v, F)[\Delta]$ is isomorphic to $\Sigma(Q)$.

It is easy to verify the next lemma.

LEMMA 1. *Let $\Delta_1, \Delta_2, \ldots, \Delta_{m+1}$ be a sequence of simplicial complexes, $F_1, F_2, \ldots, F_m$ be a sequence of faces, and $v_1, v_2, \ldots, v_m$ be a sequence of vertices, such that $\Delta_{i+1} = \text{st}(v_i, F_i)[\Delta_i]$, $1 \leq i \leq m$. Suppose, in addition, that we assume that for particular numbers j and k, $1 \leq j < k \leq m$, we have $F_k \in \Delta_j$ and $F_j \cup F_k \notin \Delta_j$. Then $v_j v_k \notin \Delta_{m+1}$.*

3. Constructing the Associahedron

Assume $n \geq 4$ and number the vertices of P_n from 0 to $n-1$ consecutively around the perimeter. Let S be the collection of all sets of consecutive integers of the form $\{i, i+1, \ldots, j\}$, where $1 \leq i \leq j \leq n-2$, excluding the set $\{1, 2, \ldots, n-2\}$. If we associate each such set with the diagonal of P_n joining vertices $i-1$ and $j+1$, we establish a bijection between the members of S and the diagonals of the n-gon.

Let Δ_1 be the boundary complex of any $(n-3)$-dimensional geometric simplex in $\mathbb{R}^{n-3}$ and number the vertices of Δ_1 from 1 to $n-2$. The members of S now correspond to certain faces of Δ_1. Order the members of S, $F_1, F_2, \ldots, F_m$, so that $i < j$ whenever $F_i \supset F_j$. Set $\Delta_{i+1} = \text{st}(v_i, F_i)[\Delta_i]$, $1 \leq i \leq m$, where v_i is not a vertex of Δ_i. Note that when F_j is subdivided, only faces containing it are lost, so that $F_{j+1}, F_{j+2}, \ldots, F_m$ are not lost, and hence the Δ_i are well defined. We remark also that the singleton sets in S correspond precisely to the original vertices of Δ_1, which need not, therefore, be subdivided.

In this manner we obtain Δ_{m+1}, which we call Δ^* for short, the vertices of which are in one-to-one correspondence with the diagonals of P_n. The fact that Δ^* is polytopal is clear since it is obtained from the boundary complex of an $(n-3)$-dimensional polytope (namely, a simplex) by a sequence of stellar subdivisions. So Δ^* is isomorphic to $\Sigma(Q_n)$ for some simplicial polytope Q_n. We will show that Δ^* is isomorphic to Σ_n, and hence that Q_n is the desired associahedron. Low values of n, say, $4 \leq n \leq 6$, can be checked directly; the procedure even works formally for $n = 3$, yielding a 0-dimensional polytope Q_3 with $\Sigma(Q_3) = \{\varnothing\} = \Sigma_3$ (see Figure 1).

The first step in showing that Δ^* is isomorphic to Σ_n will be to prove that if u and v are vertices of Δ^* corresponding to crossing diagonals of P_n, then uv is not an edge of Δ^*. For suppose u and v correspond to the sets $F = \{p, p+1, \ldots, q\}$ and $G = \{r, r+1, \ldots, s\}$ in S, respectively. If the associated diagonals cross, it is easy to see

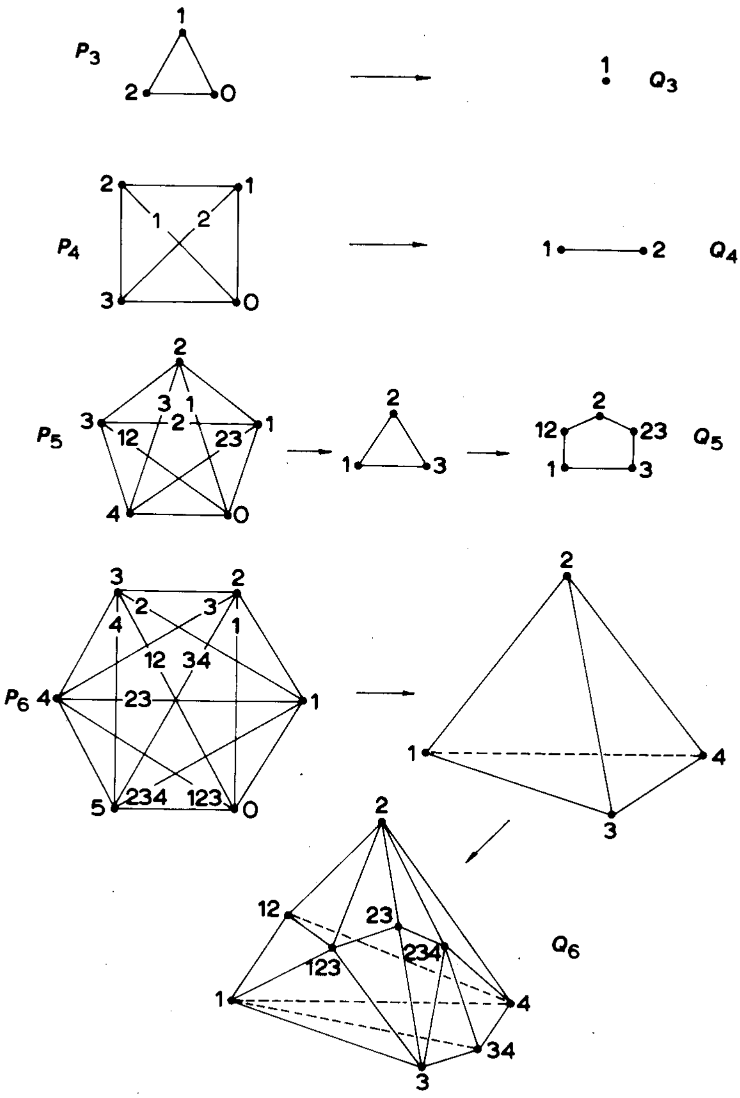

FIGURE 1

that we may assume $p < r$, $q < s$ and $r \leq q + 1$. Hence $H = \{p, p+1, \ldots, s\}$ is a set of consecutive integers containing F and G strictly. If $H = \{1, 2, \ldots, n-2\}$ then H is not a face of Δ_1, and so $uv \notin \Delta^*$ by Lemma 1. If $H \neq \{1, 2, \ldots, n-2\}$ then $H \in S$ and H is subdivided before both F and G. After its subdivision $H = F \cup G$ is no longer a face, and Lemma 1 again implies that $uv \notin \Delta^*$.

We now know that every face of Δ^* corresponds to a set of non-crossing diagonals of P_n. In particular, each facet of Δ^* represents a triangulation of the n-gon and so corresponds to a facet of Σ_n. To show the converse, it is sufficient to note that the following two properties hold for both Δ^* and Σ_n: (1) every subfacet is contained in exactly two facets; (2) between every pair of facets F and G there is a path $F = F_1$, $F_2, \ldots, F_k = G$ of facets such that F_i and F_{i+1} share a common subfacet, $i = 1, \ldots, k-1$. From this we can conclude that every facet of Σ_n corresponds to one in Δ^*. Therefore there is one facet of Δ^* for every triangulation of P_n, Δ^* is isomorphic to

Σ_n, and Q_n is the $(n-3)$-dimensional associahedron, establishing the following theorem.†

THEOREM 1. *Let Σ_n be the simplicial complex consisting of the collection of all sets of mutually non-crossing diagonals of the n-gon. Then Σ_n is realizable as the boundary complex of an $(n-3)$-dimensional simplicial polytope Q_n.*

4. THE ASSOCIAHEDRON AND GALE DIAGRAMS

In this section we describe another way to verify that Σ_n is polytopal, which will eventually lead to a realization of Q_n that geometrically reflects the symmetry of the regular n-gon. Our primary tool will be that of Gale transforms and Gale diagrams. We refer the reader to Grünbaum [6] and McMullen-Shephard [11] for definitions and explanations of any properties of Gale diagrams we may subsequently use.

Assume $n \geq 5$ and consider any convex n-gon P_n (not necessarily regular) with the vertices again numbered from 0 to $n-1$. Let X' denote this set of vertices and choose a point O in the interior of P_n such that O satisfies at least one of the following two conditions:

(1) O is in the interior of $\operatorname{conv}(X' \setminus \{x'\})$ for all $x' \in X'$.
(2) O lies on no diagonal of P_n.

Establish a Cartesian co-ordinate system for the plane such that the origin is at O. Vertex i of the n-gon can then be thought of as a vector x_i' in $\mathbb{R}^2$, $0 \leq i \leq n-1$. Because O is in the interior of P_n, there exist positive numbers λ_i, $0 \leq i \leq n-1$, such that $\sum_{i=0}^{n-1} \lambda_i x_i' = 0$. This says that O is the centroid of the vectors $\lambda_i x_i'$ and implies that the original points x_i' constitute the Gale diagram of some set of n points $X = \{x_0, x_1, \ldots, x_{n-1}\}$ in $\mathbb{R}^{n-3}$ such that $\operatorname{conv}(X)$ is a (not necessarily simplicial) $(n-3)$-dimensional polytope. We remark that some of the points in X may not be vertices of the polytope. There is a natural correspondence between the element x_i of X and the element i $(=x_i')$ of X', $0 \leq i \leq n-1$, which induces the obvious correspondence between subsets Y of X and Y' of X'.

Let Ψ be the boundary complex of this polytope. The Gale diagram has the property that for every $Y \subseteq X$ we have $Y \in \Psi$ iff O is in the relative interior of $\operatorname{conv}(X' \setminus Y')$, which we write $O \in \operatorname{relint} \operatorname{conv}(X' \setminus Y')$.

We now consider the facets, i.e. the maximal faces of Ψ. It is readily seen that $F \subseteq X$ is a facet of Ψ if $X' \setminus F'$ is the set of vertices of a triangle T or a diagonal D containing O in its relative interior. In the first case $\dim \operatorname{conv}(F) = n-4$ and $\operatorname{card} F = n-3$, and so $\operatorname{conv}(F)$ is a simplex.

In the second case $\dim \operatorname{conv}(F) = n-4$ but $\operatorname{card} F = n-2$, and so $\operatorname{conv}(F)$ is not a simplex. Suppose D has endpoints i and j. Let $G_1' = \{i+1, i+2, \ldots, j-1\}$ and $G_2' = \{j+1, j+2, \ldots, n-1, 0, 1, \ldots, i-1\}$. It is easy to check that the only proper supersets H' of $\{i, j\}$ for which $O \in \operatorname{relint} \operatorname{conv}(H')$ are the sets of the form $H' = \{i, j\} \cup H_1' \cup H_2'$, where H_i' is a non-empty subset of G_i', $i = 1,2$. This immediately implies that the boundary complex of the facet $\operatorname{conv}(F)$ is the simplicial complex $\partial \bar{G}_1 \cdot \partial \bar{G}_2$, and that with the exception of such non-simplicial facets F, every face of every dimension of Ψ corresponds to a simplex.

† We thank Gil Kalai and Micha Perles for pointing out this argument for the converse. The original argument showed by induction that the facet $F = \{1, 2, \ldots, n-2\} \setminus \{j\}$ of Δ_1 was ultimately subdivided into $c_j c_{n-j-1}$ facets of Δ^*, $1 \leq j \leq n-2$. Then the identity $\sum_{j=1}^{n-2} c_j c_{n-j-1} = c_{n-1}$ verifies that all of the facets of Σ_n are present in Δ^*, offering a nice geometric manifestation of the Catalan recurrence relation.

To construct the associahedron, we begin by subdividing each non-simplicial facet F in a manner analogous to stellar subdivision by removing F and adding all faces of the form $\{v\} \cup G$, where $G \in \partial \bar{G}_1 \cdot \partial \bar{G}_2$. When this is done for every such F, Ψ is transformed into a simplicial complex Ψ_1. The same argument as for stellar subdivisions shows that Ψ_1 is polytopal; we can place a point v just 'above' the centroid of conv(F) and take the convex hull. Note that apart from the non-simplicial facets of Ψ, no other face of Ψ is lost.

A proper subset of vertices of X' will be called *consecutive* if it is a set of consecutive integers, mod n. Consider any diagonal of P_n not containing the origin. When extended, the diagonal determines two open half-planes, one of which contains O. Associate with the diagonal the set F' of consecutive vertices in the opposite open half-space. Let S' be the collection of all subsets of X' derived in this way. We then have a bijection between the members of S' and the diagonals of P_n not containing the origin. We observe that if $F' \in S'$, then every consecutive subset of F' is also in S'. Furthermore, if G_i' is one of the two consecutive sets associated with a diagonal containing O as previously described, then every proper consecutive subset of G_i' is in S'.

By the property of Gale diagrams, every member of S' corresponds to a face of Ψ, and hence of Ψ_1. Note in particular that the singleton sets in S' correspond precisely to the original vertices of Ψ. Order the faces of Ψ_1 associated with the members of S', $F_1, F_2, \ldots, F_r$, so that $i<j$ whenever $F_i \supset F_j$, and set $\Psi_{i+1} = \mathrm{st}(v_i, F_i)[\Psi_i]$, $1 \leqslant i \leqslant r$. Once again we obtain a polytopal simplicial complex $\Psi^* = \Psi_{r+1}$, the vertices of which are in one-to-one correspondence with the diagonals of the n-gon (see Figure 2). The argument showing that Ψ^* is isomorphic to Σ_n will parallel the discussion of the previous section.

Suppose u and v are vertices of Ψ^* associated with crossing diagonals D and E, respectively. If D and E both contain O, then u and v were introduced to triangulate two distinct non-simplicial facets of Ψ. Hence $uv \notin \Psi_1$ and so $uv \notin \Psi^*$. Suppose $O \in D$ but $O \notin E$. The only way we could have $uv \in \Psi^*$ is if $F \in \mathrm{lk}_{\Psi_1} u$, where F is the face subdivided by v. But $\mathrm{lk}_{\Psi_1} u = \partial \bar{G}_1 \cdot \partial \bar{G}_2$, where G_1' and G_2' are the two consecutive sets defined by the two open half-planes associated with D. Hence $F' \subseteq G_1'$ or $F' \subseteq G_2'$ and in either case D and E cannot cross.

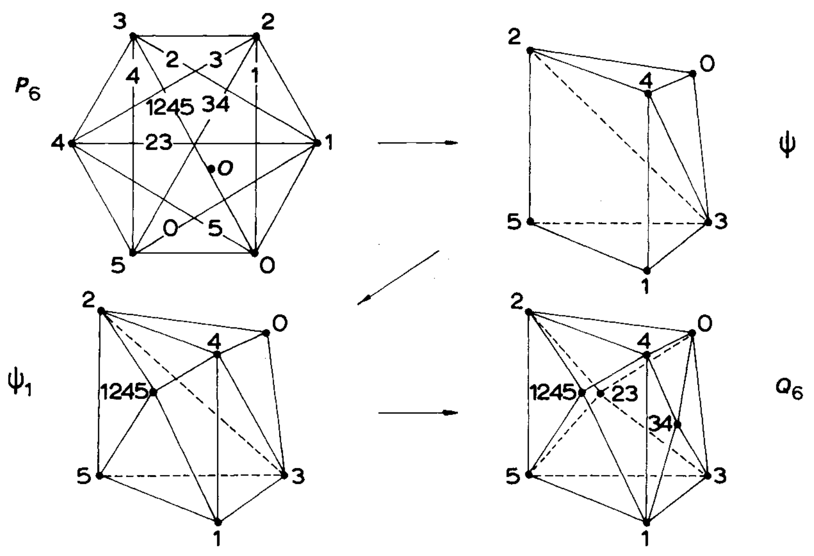

FIGURE 2

Finally, suppose neither D nor E contains O. If u and v correspond to consecutive sets F' and G', respectively, then one can verify that $H' = F' \cup G'$ is a set of consecutive vertices strictly containing both F' and G'. If H is not a face of Ψ_1 then $uv \notin \Psi^*$ by Lemma 1. If H is a face of Ψ_1 then H is a face of Ψ and it is easy to see that H' must also be in S'. Hence H is subdivided before both F and G. When H is subdivided, then $F \cup G$ is no longer a face, so once again $uv \notin \Psi^*$.

We now know that every facet of Ψ^* corresponds to a facet of Σ_n. The proof of the converse is identical to the previous argument for Δ^*. Hence Σ_n is isomorphic to Ψ^* and thus to $\Sigma(Q_n)$ for some simplicial $(n-3)$-polytope Q_n. The above construction includes the construction of the previous section as a special case. One need only choose O to be suitably near a point in the relative interior of the edge joining 0 and $n-1$.

Since the boundary complex of any $(n-3)$-polytope with at most n vertices can be refined to that of a simplicial $(n-3)$-polytope with n vertices, and since every such simplicial polytope has a Gale digram consisting of a convex n-gon with origin O in its interior satisfying condition (2), we have the following result.

THEOREM 2. *For any $(n-3)$-polytope P with at most n vertices, there exists a refinement of the boundary complex that is isomorphic to Σ_n. Moreover, if P is simplicial, the refinement is achievable by a sequence of stellar subdivisions.*

5. SYMMETRICAL REALIZATIONS

We will now determine a realization of Q_n that geometrically reflects the symmetry of the regular n-gon. Specifically, we will construct Q_n in such a way that its symmetry group is isomorphic to the dihedral group D_n. Suppose P_n is a regular n-gon with vertex j having co-ordinates $(\cos j\theta, \sin j\theta)$, $0 \le j \le n-1$, where $\theta = 2\pi/n$. The dihedral group is generated by elements g_1 and g_2, where $g_1(j) = j + 1 (\mathrm{mod}\, n)$ and $g_2(j) = n - j (\mathrm{mod}\, n)$, $0 \le j \le n-1$.

Because in the above situation the origin O is the centroid of the vertices of P_n, we in fact have a Gale diagram that is a Gale transform of some $(n-3)$-polytope R_n if $n \ge 5$. Moreover, R_n has n vertices $x_0, x_1, \ldots, x_{n-1}$ which are in one-to-one correspondence with the vertices $0, 1, \ldots, n-1$ of the n-gon.

To find the co-ordinates of the vertices of R_n, we first consider the set of n non-zero vectors $\{u^0, u^1, \ldots, u^{\lfloor n/2 \rfloor}, v^1, v^2, \ldots, v^{\lfloor (n-1)/2 \rfloor}\}$, where $\lfloor \cdot \rfloor$ denotes the integer round-down function, defined by

$$\begin{aligned} u^k &= (u_0^k, u_1^k, \ldots, u_{n-1}^k), && 0 \le k \le \lfloor n/2 \rfloor, \\ u_j^k &= \cos kj\theta, && 0 \le j \le n-1, \\ v^k &= (v_0^k, v_1^k, \ldots, v_{n-1}^k), && 1 \le k \le \lfloor (n-1)/2 \rfloor, \\ v_j^k &= \sin kj\theta, && 0 \le j \le n-1. \end{aligned}$$

Note in particular that

$$\begin{aligned} u^0 &= (1, 1, \ldots, 1), \\ u^1 &= (\cos 0\theta, \cos 1\theta, \ldots, \cos(n-1)\theta), \\ v^1 &= (\sin 0\theta, \sin 1\theta, \ldots, \sin(n-1)\theta), \end{aligned}$$

and

$$u^{\lfloor n/2 \rfloor} = (1, -1, 1, -1, \ldots, -1) \qquad \text{if } n \text{ is even.}$$

Using the fact that $\sum_{j=0}^{n-1} \omega^{mj} = 0$ if n does not divide m, where ω is the complex nth root of unity $\cos\theta + i\sin\theta$, and other elementary trigonometric identities, it is easy to check that we have a set of n non-zero mutually orthogonal vectors, one of which is the vector $(1, 1, \ldots, 1)$.

If we list vectors u^1 and v^1 as the rows of a $2 \times n$ matrix, the columns provide the co-ordinates of the regular n-gon. This implies that if we list all of our vectors except $u^0 = (1, 1, \ldots, 1)$, u^1 and v^1 as the rows of an $(n-3) \times n$ matrix, the columns of the matrix provide the co-ordinates of $x_0, x_1, \ldots, x_{n-1}$, respectively. Thus we may take

$$x_j = \left(\cos 2j\theta, \sin 2j\theta, \ldots, \cos\left(\frac{n-1}{2}\right)j\theta, \sin\left(\frac{n-1}{2}\right)j\theta\right), \qquad 0 \leq j \leq n-1, \quad \text{if } n \text{ is odd},$$

and

$$x_j = \left(\cos 2j\theta, \sin 2j\theta, \ldots, \cos\left(\frac{n-2}{2}\right)j\theta, \sin\left(\frac{n-2}{2}\right)j\theta, (-1)^j\right) \quad 0 \leq j \leq n-1, \quad \text{if } n \text{ is even}.$$

In the former case R_n is a cyclic $(n-3)$-polytope, and in the latter case R_n is the projection of a cyclic $(n-2)$-polytope.

Suppose n is odd. Define g_1' to be the $(n-3) \times (n-3)$ matrix $\operatorname{diag}(B_2, B_3, \ldots, B_{\lfloor (n-1)/2 \rfloor})$ where B_k is the 2×2 block

$$\begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}.$$

If n is even, define g_1' to be the $(n-3) \times (n-3)$ matrix $\operatorname{diag}(B_2, B_3, \ldots, B_{\lfloor (n-2)/2 \rfloor}, -1)$ with the 2×2 blocks B_k defined in the same way. Whatever the parity of n, define g_2' to be the $(n-3) \times (n-3)$ matrix $\operatorname{diag}(1, -1, \ldots, (-1)^{n-2})$. It is easy to check that g_1' and g_2' generate the group of orthogonal symmetries of R_n isomorphic to the dihedral group, where $g_1'(x_j) = x_{j+1 (\text{mod } n)}$ and $g_2'(x_j) = x_{n-j (\text{mod } n)}$, $0 \leq j \leq n-1$.

It is also straightforward to verify that every face of R_n to be subdivided is mapped by any element of the group onto another such face, and that centroids are mapped onto centroids. Therefore all the necessary subdivisions to the boundary complex of R_n can be carried out geometrically in such a way that the group is also the group of symmetries of the resulting associahedron Q_n. For example, if a face F with centroid y is to be subdivided via a vertex z, choose $z = (1+\varepsilon)y$, where ε is a suitably small positive number taken to be the same for all faces in the orbit of F (see Figure 3).

6. The f-vector and h-vector of the Associahedron

In this section we investigate the number of j-dimensional faces f_j, $0 \leq j \leq n-4$, of the $(n-3)$-dimensional polytope Q_n. Of course, we know that f_j equals the number of ways of choosing a set of $j+1$ mutually non-crossing diagonals of the convex n-gon P_n. In particular, $f_{n-4} = c_{n-1}$. The *f-vector* of Q_n is the vector $f(Q_n) = (f_{-1}, f_0, f_1, \ldots, f_{n-4})$, where we take $f_{-1} = 1$ by convention.

The h-vector of Q_n is defined by $h(Q_n) = (h_0, h_1, \ldots, h_{n-3})$, where

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{n-j-3}{n-i-3} f_{j-1}, \qquad 0 \leq i \leq n-3, \tag{1}$$

and the f-vector can be recovered from the h-vector by

$$f_{j-1} = \sum_{i=0}^{j} \binom{n-i-3}{n-j-3} h_i, \qquad 0 \leq j \leq n-3. \tag{2}$$

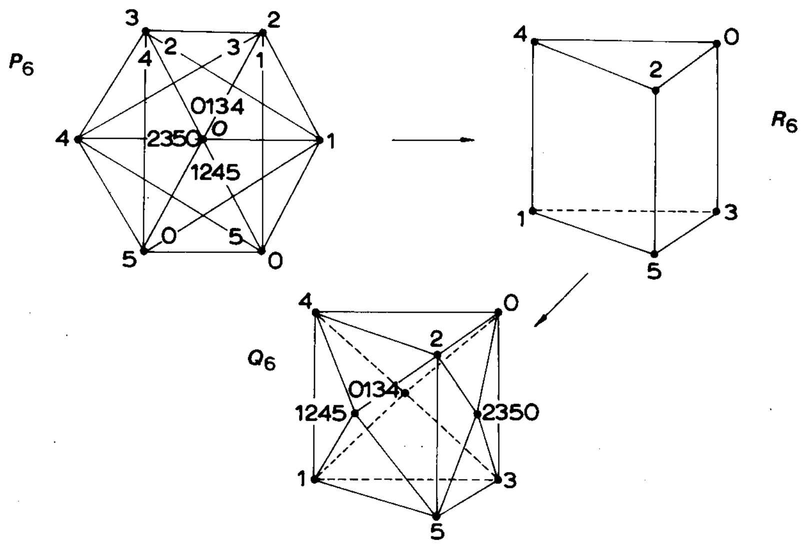

FIGURE 3

See, for example, McMullen-Shephard [11] where our h_i is their $g_{i-1}^{(d)} = g_{i-1}^{(n-3)}$. Past experience has shown that the h-vector is often more tractable than the f-vector, and this turns out to be the case here too.

THEOREM 3. *For the associahedron Q_n,*

$$f_{j-1} = \frac{1}{n-1}\binom{n-3}{j}\binom{n+j-1}{j+1}, \qquad 0 \leqslant j \leqslant n-3,$$

and

$$h_i = \frac{1}{n-1}\binom{n-3}{i}\binom{n-1}{i+1}, \qquad 0 \leqslant i \leqslant n-3.$$

PROOF. The first formula is that of Kirkman [9] and Cayley [2], and the second follows from (1). □

The fact that $h_i = h_{n-3-i}$ is a manifestation of the *Dehn–Sommerville equations* (see [6,11]) which hold for any triangulated sphere.

Our next objective is to describe the components of the h-vector combinatorially. Fix any triangulation T of P_n, $n \geqslant 4$. We will color each of its diagonals either red or green, according to the following method. Choose a diagonal D and remove it, leaving a 'hole' in the shape of a quadrilateral. There are exactly two diagonals of P_n that are also diagonals of the quadrilateral. One is D; call the other D'. Notice that D and D' are crossing, and in particular share no common endpoint. Labeling the vertices of the n-gon as before, traverse them in the order $0, 1, \ldots, n-1$, noting for which of D, D' you encounter an endpoint first. If D is met first, color D green; otherwise, color it red.

We now observe that given any set of mutually non-crossing diagonals of P_n (not necessarily a triangulation) there is exactly one way to complete the set to a triangulation T such that every newly added diagonal is green in T. For suppose we have not yet completed the set to a triangulation. Then there is at least one convex m-gon, $m \geqslant 4$, in this subdivision, bounded by diagonals from the set and sides of P_n. Let its vertices be $\{i_1, i_2, \ldots, i_m\}$, where $i_1 < i_2 < \cdots < i_m$. No new green diagonal in a triangulation extending the given set can have i_m as an endpoint; hence any such triangulation must contain the diagonal joining i_1 and i_{m-1}. By repeating this argument, the uniquely determined T is constructed.†

THEOREM 4. *For the associahedron Q_n, h_i equals the number of triangulations of P_n having exactly i red diagonals.*

PROOF. Let g_i be the number of triangulations with exactly i red diagonals. Let F be any set of j mutually non-crossing diagonals of P_n. There is exactly one way to complete F to a triangulation so that all of the $n-j-3$ new diagonals are green. This means we can count the number of such F by counting the number of ways we can choose a triangulation with exactly i red diagonals, $i \leq j$, and then remove $n-j-3$ of the $n-i-3$ green diagonals. Thus

$$f_{j-1} = \sum_{i=0}^{j} \binom{n-i-3}{n-j-3} g_i, \qquad 0 \leq j \leq n-3.$$

Formulas (1) and (2) immediately imply $g_i = h_i$, $0 \leq i \leq n-3$. □

The Dehn–Sommerville equations are a consequence of being able to interchange the colors green and red. For a dual version of this type of counting argument, see Brøndsted [1].

The components of $h(Q_n)$ can be interpreted in terms of some of the many problems isomorphic to that of triangulating an n-gon [5]:

(1) Consider all ways of completely parenthesizing a sequence of $n-1$ symbols using $n-2$ pairs of parentheses. Then h_i equals the number of parenthesizations containing exactly i internal groups of left (respectively right) parentheses. Modifying the technique discussed in [4] to obtain the formula for the Catalan numbers, one can exploit this isomorphism to derive the formula for h_i directly, from which the formula for f_{j-1} is an easy corollary.
(2) Consider all sequences of length $2n-4$ composed of $n-2$ zeros and $n-2$ ones, such that at no position along the sequence have you encountered more zeros than ones. Then h_i equals the number of sequences with $i+1$ blocks of ones.
(3) Consider all paths from the point $(0,0)$ to the point $(n-2, n-2)$ in the Cartesian plane, where only unit steps upward and to the right are allowed, and where you must never pass through a point above the line joining $(0,0)$ and $(n-2, n-2)$. Then h_i equals the number of paths with i changes of direction from upward to right.
(4) Consider all rooted, planar, trivalent trees with one root and $n-1$ other nodes of degree 1. Then h_i equals the number of trees with i branchings to the left (respectively right).
(5) Consider all rooted, planar trees with one root and $n-1$ other nodes, whether of degree one or not. Let us say there are $k-2$ branchings at a node of degree $k \geq 3$. Then h_i equals the number of trees with a total of i branchings.

Notice the appearance of the Dehn–Sommerville equations again in (1) and (4).

7. CONCLUDING REMARKS

We wish to mention another polytope associated with the triangulations of the n-gon. Dantzig, Hoffman and Hu [3] have shown how to describe a polytope by linear

† This argument, suggested by a referee, is essentially isomorphic to our original argument but avoids recasting the problem in terms of parenthesizing a sequence of $n-1$ symbols.

equations in non-negative variables, the vertices of which correspond to the triangulations of the n-gon and the facets of which correspond to the diagonals. The dual of this polytope has therefore one vertex for every diagonal and one facet for every triangulation. But this dual is not isomorphic to Q_n; in general, it is higher dimensional. It is true, however, that adjacent triangulations correspond to adjacent facets, although the converse does not hold.

Given any d-dimensional convex polytope P, one might consider the set Σ of all subdivisions of P, partially ordered by refinement, and ask whether Σ is realizable as the boundary complex of some simplicial convex polytope Q of dimension $n-d-1$, with facets of Q corresponding to triangulations of P. As we have shown, this is true if $d=2$. It also turns out to be true if $n \leqslant d+3$, but fails in general (for example, when $d=3$ and $n=7$). Nevertheless, there always exists a nice $(n-d-2)$-dimensional spherical complex of some, but not necessarily all, subdivisions of the polytope [10].

Note added in proof: I. M. Gel'fand, A. V. Zelevinskij and M. M. Kapranov have shown this complex to be polytopal if the original polytope is rational.

Acknowledgments

I wish to thank Mark Haiman, Gil Kalai, Micha Perles and an anonymous referee for their helpful suggestions and comments. Research supported, in part, by NSF grants MCS-8201653 and DMS-8504050, and an Alexander von Humboldt Foundation Fellowship.

References

1. A. Brøndsted, *An Introduction to Convex Polytopes*, Springer-Verlag, New York, 1983.
2. A. Cayley, On the partitions of a polygon, *Proc. Lond. Math. Soc.* (1) **22** (1890–1891), 237–262.
3. G. B. Dantzig, A. J. Hoffman and T. C. Hu, Triangulations (tilings) and certain block triangular matrices, *Math. Programm.* **31** (1985), 1–14.
4. H. Dörrie, Euler's problem of polygon division, *100 Great Problems of Elementary Mathematics* (transl. D. Antin), Dover, New York, 1965, pp. 21–27.
5. M. Gardner, Catalan numbers, *Time Travel and Other Mathematical Bewilderments,* W. H. Freeman, New York, 1988, pp. 253–266.
6. B. Grünbaum, *Convex Polytopes,* Interscience, London, 1967.
7. M. Haiman, Constructing the associahedron, manuscript.
8. D. Huguet and D. Tamari, La structure polyèdrale des complexes de parenthésages, *J. Comb., Inf. & Syst. Sci.*, **3** (1978), 69–81.
9. T. P. Kirkman, On the k-partitions of the r-gon and r-ace, *Phil. Trans. R. Soc. Lond.*, **147** (1857), 217–272.
10. C. W. Lee, Some notes on triangulating polytopes, *Proceedings, 3. Kolloquium über Diskrete Geometrie,* Institut für Mathematik, Universität Salzburg, Salzburg, Austria, May 1985, pp. 173–181.
11. P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture,* Cambridge University Press, 1971.
12. M. A. Perles, List of problems, Conference on Convexity, Oberwolfach, Federal Republic of Germany, July 1984.

Received 16 January 1985 and in revised form 24 March 1988

Carl W. Lee
Department of Mathematics,
University of Kentucky,
Lexington, Kentucky 40506, U.S.A.